§3.8: Newton’s Method: Approximate Zeros

Idea: Mathematica

Purpose: There is not always a formula to find zero’s of an equation.

Archimedes wanted to cut a sphere where the volume on one side is twice the other. This example leads to solving the equation $3x^3 - 9x + 2 = 0$.

Example: $\cos(x) = x$

Example: $x^5 - 9x^4 + 2x^3 - 5x^2 + 17x - 8 = 0$

Idea for Newton’s Method:

1. Find a place to start (near solution)
2. Find the tangent line
   \[ y - f(x_i) = f'(x_i)(x - x_i) \]
3. Find the x-intercept
   \[ 0 - f(x_i) = f'(x_i)(x - x_i) \]
4. Solve for $x_2$
5. Repeat

This process will produce the general term

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

Does not always work:

Example: $f(x) = x^{\frac{1}{3}} - \sqrt[3]{x}$

\[ f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \]

\[ f'(x) = \frac{1}{3\sqrt[3]{x}} \]
Example: \( f(x) = x^3 - 2x - 1 \) has a zero between 1 & 2.

Use Newton's method to approximate the value to 6 decimal places.

\[
\begin{align*}
\chi_0 &= 1.5 \\
\chi_1 &= 1.5 - \frac{f(1.5)}{f'(1.5)} \\
f(x) &= x^3 - 2x - 1 \\
f'(x) &= 3x^2 - 2 \\
\chi_1 &= 1.6315789 \\
\chi_2 &= 1.6181836 \\
\chi_3 &= 1.6180333
\end{align*}
\]

Example: \( f(x) = x^3 - x^2 - 1 \) start at \( x=2 \).

\[
\begin{align*}
\chi_0 &= 2 \\
f(x) &= x^3 - x^2 - 1 \\
f'(x) &= 3x^2 - 2x \\
\chi_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
\chi_1 &= 2 - \frac{3}{8} = 1.5 \\
\chi_2 &= 1.5 - \frac{f(1.5)}{f'(1.5)} =
\]

Example: Solve \( \cos(x) = x \)

\[
\begin{align*}
f(x) &= \cos(x) - x \\
f'(x) &= -\sin(x) - 1 \\
\chi_{n+1} &= \chi_n - \frac{f(x_n)}{f'(x_n)}
\end{align*}
\]

Example: Use Newton's method to approximate \( \sqrt[3]{30} \)

\[
\begin{align*}
\chi &= \sqrt[3]{30} \\
f(x) &= x^3 - 30 \\
f'(x) &= 3x^2 \\
\chi_0 &= 3 \\
\chi_1 &= 3 - \frac{f(3)}{f'(3)} = 3 - \frac{(-3)}{27} \\
\chi_2 &= 3 - \frac{f(3^{1/3})}{f'(3^{1/3})}
\end{align*}
\]